On the Schrödinger Equation for the Minisuperspace Models

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Abstract

We obtain a time-dependent Schrödinger equation for the Friedmann - Robertson - Walker (FRW) model interacting with a homogeneous scalar matter field. We show that for this purpose it is necesary to include an additional action invariant under the reparametrization of time. The last one does not change the equations of motion of the system, but changes only the constraint which at the quantum level becomes time-dependent Schrödinger equation. The same procedure is applied to the supersymmetric case and the supersymmetric quantum constraints are obtained, one of them is a square root of the Schrödinger operator.

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One of the most important questions in quantum cosmology is that of identifying a suitable time parameter [1] and a time-dependent Wheeler-DeWitt equation [2, 3]. The main peculiarity of the gravity theory is the presence of non-physical variables (gauge variables) and constraints [3, 4, 5, 6]. They arise due to the general coordinate invariance of the theory. The conventional Wheeler-DeWitt formulation gives a time independent quantum theory [7]. The canonical quantization of the minisuperspace approximation [8] has been used to find results in the hope, that they would illustrate the behaviour of general relativity [9]. In the minisuperspace models [2, 7] there is a residual invariance under reparametrization of time (world-line symmetry). Due to this fact the equation that governs the quantum behaviour of these models is the Schrödinger equation for states with zero energy. On the other hand, supersymmetry transformations are more fundamental than time translations (reparametrization of time) in the sense, that these ones may be generated by anticommutators of the supersymmetry generators. The recent introduction of supersymmetric minisuperspace models has led to the square root equations for states with zero energy [10, 11, 12]. The structure of the world-line supersymmetry or the world-line supersymmetry transformations has led to the zero Hamiltonian phenomena [2, 6, 12]. Investigations of the time evolution problem for such quantum systems have been carried in two directions: the cosmological models of gravity have been quantized by reducing the phase space degrees of freedom [13, 14, 15, 16, 17] and with the help of the WKB approach [18, 19].

In this work we obtain a time-dependent Schrödinger equation for the homogeneous cosmological models. In our approach this equation arises due to an additional action invariant under reparametrization. The last one does not change the equations of motion, but the constraint which becomes time-dependent Schrödinger equation. In the case of the supersymmetric minisuperspace model we obtain the supersymmetric constraints, one of them is a square root of time-dependent Schrödinger equation.

We begin by considering an homogeneous and isotropic metric defined by

$$ds^2 = -N^2(t)dt^2 + R^2(t)d\Omega_3^2, \tag{1}$$

where the only dynamical degree of freedom is the scale factor $R(t)$. The lapse function $N(t)$, being a pure gauge variable, is not dynamical. The quantity $d\Omega_3^2$ is the standard line element on the unit three-sphere. We shall set $c = \hbar = 1$. The pure gravitational action corresponding to the metric (1) is

$$S_g = \frac{6}{\kappa^2}\int \left(-\frac{R\dot{R}^2}{2N} + \frac{1}{2}kNR\right) dt, \tag{2}$$

where $k = 1, 0, -1$ corresponds to a closed, flat or open space. $\kappa^2 = 8\pi G_N$, where G_N is the Newton's constant of gravity, and the overdot denotes differentiation with respect to t. The action (2) preserves the invariance under the time reparametrization

$$t' \to t + a(t), \tag{3}$$

if the transformations of $N(t)$ and $R(t)$ are

$$\delta R = a\dot{R} \qquad\qquad \delta N = \dot{a}N + a\dot{N} \tag{4}$$

that is, $R(t)$ transforms as a scalar and $N(t)$ as a one-dimensional vector, and its dimensionality is the inverse of $a(t)$.

So, we consider the interacting action for the homogeneous real scalar matter field $\phi(t)$ and the scale factor $R(t)$. This action has the form

$$S_m = \int \left(\frac{R^3\dot{\phi}^2}{2N} - NR^3V(\phi) \right) dt. \tag{5}$$

This action remains invariant under the local transformation (3), if in addition to the transformation law for $R(t)$ and $N(t)$ in (4), the field $\phi(t)$ transforms as a scalar; $\delta\phi = a\dot{\phi}$.

Thus, our system is described by the full action

$$S = S_g + S_m = \int \left(-\frac{3R\dot{R}^2}{\kappa^2 N} + \frac{R^3\dot{\phi}^2}{2N} + \frac{3kNR}{\kappa^2} - NR^3V(\phi) \right) dt. \tag{6}$$

Now, we shall consider the Hamiltonian analysis of this action. The canonical momenta for the variables R and ϕ are given, respectively, by

$$P_R = \frac{\partial L}{\partial \dot{R}} = -\frac{6R\dot{R}}{\kappa^2 N}, \qquad P_\phi = \frac{R^3\dot{\phi}}{N}. \tag{7}$$

Their canonical Poisson brackets are defined as

$$\{R, P_R\} = 1, \qquad\qquad \{\phi, P_\phi\} = 1. \tag{8}$$

The canonical momentum for the variable $N(t)$ is

$$P_N \equiv \frac{\partial L}{\partial \dot{N}} = 0, \tag{9}$$

this equation merely constrains the variable $N(t)$ (primary constraint). The canonical Hamiltonian can be calculated in the usual way, it has the form $H_c = NH_0$, then the total Hamiltonian is

$$H_T = NH_0 + u_N P_N, \tag{10}$$

where u_N is the Lagrange multiplier associated to the constraint $P_N = 0$ in (9), and H_0 is the Hamiltonian written as

$$H_0 = \left(-\frac{\kappa^2 P_R^2}{12R} + \frac{\pi_\phi^2}{2R^3} - \frac{3kR}{\kappa^2} + R^3V(\phi) \right). \tag{11}$$

The time evolution of any dynamical variables is generated by (10). For the compatibility of the constraint the Eq. (9) and the dynamics generated by the total Hamiltonian of Eq. (10), the following equation must hold

$$H_0 = 0, \tag{12}$$

which constrains the dynamics of our system. So, we proceed to the quantum mechanics from the above classical system. We introduce the wave function of the Universe ψ. The constraint equation (12) must be imposed on the states

$$H_0 \psi = 0. \tag{13}$$

This constraint nullifies all the dynamical evolution generated by the total Hamiltonian (10). A commutator of any operator and the total Hamiltonian becomes zero, if it is evaluated for the above constrained states. The disappearence of time seems disappointing, however, it is a proper consequence of the invariance of general coordinate transformation in general relativity. The equation (9) merely says, that the wave function ψ does not depend on the lapse function $N(t)$. Therefore, we expect that the equation in (13) may contain any information of dynamics. In quantum cosmology the constraint (13) is well-known as the Wheeler-DeWitt equation (time-independent Schrödinger equation).

In order to get a time-dependent Schrödinger equation we shall regard the following invariant action

$$S_r = -\frac{1}{\kappa^3} \int R^3 P_T \left(-\frac{dT(t)}{dt} + N(t) \right) dt, \tag{14}$$

where (T, P_T) is a pair of dynamical variables, P_T is the momentum conjugate to T. This action is invariant under reparametrization (3), if P_T and T transform as

$$\delta P_T = a \dot{P}_T \qquad\qquad \delta T = a \dot{T}, \tag{15}$$

and N, R as in (4).

So, adding the action (14) to the action (6) we have the total invariant action $\tilde{S} = S_g + S_m + S_r$. In the first order form we get

$$\tilde{S} = \int \left\{ \dot{R} P_R + \dot{\phi} P_\phi - N H_0 + \frac{R^3 P_T}{\kappa^3} \left(\frac{dT(t)}{dt} - N(t) \right) \right\} dt. \tag{16}$$

We shall proceed with the canonical quantization of the action (16). We define the canonical momenta π_T and π_{P_T} corresponding to the variables T and P_T, respectively. We get

$$\pi_T \equiv \frac{\partial \tilde{L}}{\partial \dot{T}} = \frac{R^3}{\kappa^3} P_T, \qquad\qquad \pi_{P_T} \equiv \frac{\partial \tilde{L}}{\partial \dot{P}_T} = 0, \tag{17}$$

leading to the constraints

$$\Pi_1 \equiv \pi_T - \frac{R^3}{\kappa^3} P_T = 0, \qquad\qquad \Pi_2 \equiv \pi_{P_T} = 0. \tag{18}$$

So, we define the matrix C_{AB}, $(A, B = 1, 2)$ as a Poisson brackets between the constraints $C_{AB} = \{\Pi_A, \Pi_B\}$. Then, we have the following non-zero matrix elements

$$\{\Pi_1, \Pi_2\} = -\frac{R^3}{\kappa^3}, \tag{19}$$

with their inverse matrix elements $(C^{-1})^{1,2} = \frac{\kappa^3}{R^3}$. The Dirac's brackets $\{,\}^*$ are defined by

$$\{f, g\}^* = \{f, g\} - \{f, \Pi_A\} C^{AB} \{\Pi_B, g\}. \tag{20}$$

The result of this procedure leads to the non-zero Dirac's bracket relation

$$\{T, P_T\}^* = \frac{\kappa^3}{R^3}. \tag{21}$$

Then, the canonical Hamiltonian is

$$\tilde{H}_c = N \left(\frac{R^3}{\kappa^3} P_T + H_0 \right), \tag{22}$$

where the Hamiltonian constraint corresponding to the action (16) is

$$\tilde{H} = \frac{R^3}{\kappa^3} P_T + H_0. \tag{23}$$

At the quantum level the Dirac's brackets become commutators

$$[T, P_T] = i\{T, P_T\}^* = i\frac{\kappa^3}{R^3}. \tag{24}$$

So, taking the momentum P_T corresponding to T as

$$P_T = -i\frac{\kappa^3}{R^3}\frac{\partial}{\partial T}, \tag{25}$$

the quantum constraint (23) becomes quantum equation on the wave function ψ

$$i\frac{\partial}{\partial T}\psi(T, R, \phi) = H\left(-i\frac{\partial}{\partial R}, -i\frac{\partial}{\partial \phi}, R, \phi\right)\psi. \tag{26}$$

Explicitly, we have

$$i\frac{\partial \psi}{\partial T} = \left[\frac{\kappa^2}{12}\frac{1}{R^2}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) - \frac{3kR}{\kappa^2} - \frac{1}{2R^3}\frac{\partial^2}{\partial \phi^2} + R^3 V(\phi)\right]\psi. \tag{27}$$

This equation is the time-dependent Schrödinger equation for minisuperspace.

Equations of motion are obtained by demanding that the action $\tilde{S} = S_g + S_m + S_r$ is extremal, *i.e.* the functional derivatives of $\tilde{S}$ must be zero

$$\frac{\delta \tilde{S}}{\delta R} = \frac{\delta S_g}{\delta R} + \frac{\delta S_m}{\delta R} + \frac{\delta S_r}{\delta R} = 0. \tag{28}$$

As a consequence of the equation of motion

$$\frac{\delta \tilde{S}}{\delta P_T} = \frac{\delta S_r}{\delta P_T} = \frac{R^3}{\kappa^3}(\dot{T} - N) = 0, \tag{29}$$

the last term in (28), $\frac{3R^2}{\kappa^3} P_T(\dot{T} - N)$ dissapears and, in fact, inclusion in S of an additional invariant action S_r does not change the equations of motion axcept the equation $\frac{\delta \tilde{S}}{\delta N} = 0$, which is the constraint (23).

In the case of the Arnowit-Deser-Misner (ADM) formalism [3] the additional term (14) can be written in the following invariant form

$$\begin{aligned} S_{(d=4)} &= -\frac{1}{\kappa^3} \int \sqrt{-g} P_T(-n^\mu \partial_\mu T + 1) d^4x \\ &= -\frac{1}{\kappa^3} \int N h^{1/2} P_T \left(-\frac{\partial_0 T}{N} - \frac{N^i \partial_i T}{N} + 1 \right) dt d^3x \\ &= -\frac{1}{\kappa^3} \int h^{1/2} P_T (-\partial_0 T - N^i \partial_i T + N) dt d^3x. \end{aligned} \tag{30}$$

According to the ADM prescription [3] of classical general relativity, one considers a slicing of the space-time by a family of space-like hypersurfaces labeled by a parameter t. This parameter can be thought of as a time coordinate, so that each slice is identified by the relation $t = const$. The remaining three spatial coordinates, x^i, determine a coordinatization of each slice. The space-time metric $g_{\mu\nu}$ is decomposed into shift N^i, lapse N functions and the three-metric of the slice h_{ij}. In the action (30), $h = det h_{ij}$, $\sqrt{-g} = N\sqrt{h}$ and n^μ $(n^\mu n_\mu = -1)$ is the unit normal vector to hypersurface $t = const$ with components $n_\mu = (-N, 0, 0, 0)$ and $n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right)$. In the case of the homogeneous metric (1) the shift vector is $N^i = 0$ and $h^{1/2} = R^3$.

So, if we consider the four-dimensional gravity interacting with a scalar matter field and the invariant additional term (30), then after applying the (ADM) $(3+1)$ formalism for the FRW model we get

$$\begin{aligned} S &= -\frac{1}{2\kappa^2} \int \sqrt{-g} R d^4x - \int \sqrt{-g} \left[\frac{(\partial_\mu \phi)^2}{2} + V(\phi) \right] d^4x \\ &- \frac{1}{\kappa^3} \int \sqrt{-g} P_T (n^\mu \partial_\mu T - 1) d^4x = \int \left[\left(-\frac{3R\dot{R}^2}{2N\kappa^2} + \frac{3}{\kappa^2} kNR \right) + \right. \\ &+ \left. \frac{R^3 \dot{\phi}^2}{2N} - NR^3 V(\phi) \right] dt + \frac{1}{\kappa^3} \int P_T R^3 \left(\frac{dT(t)}{dt} - N(t) \right) dt. \end{aligned} \tag{31}$$

In particular, choosing the gauge $N = 1$, then $T = t$ and we obtain the so-called cosmic time, on the other hand, if we take $N = \frac{R}{\kappa}$ we get the conformal time gauge.

In order to obtain a superfield formulation of the action (6) the transformation of the time reparametrization (3) must be extended to the $n = 2$ local conformal time supersymmetry (LCTS) $(t, \eta, \bar{\eta})$ [20, 21]. These (LCTS) transformations can be written as

$$\begin{aligned}
\delta t &= \mathbb{L}(t,\theta,\bar{\theta}) + \frac{1}{2}\bar{\theta} D_{\bar{\theta}} \mathbb{L}(t,\theta,\bar{\theta}) - \frac{1}{2}\theta D_{\theta} \mathbb{L}(t,\theta,\bar{\theta}), \\
\delta\theta &= \frac{i}{2} D_{\bar{\theta}} \mathbb{L}(t,\theta,\bar{\theta}), \qquad \delta\bar{\theta} = -\frac{i}{2} D_{\theta} \mathbb{L}(t,\theta,\bar{\theta}),
\end{aligned} \tag{32}$$

with the superfunction $\mathbb{L}(t,\theta,\bar{\theta})$ defined by

$$\mathbb{L}(t,\theta,\bar{\theta}) = a(t) + i\theta\bar{\beta}'(t) + i\bar{\theta}\beta'(t) + b(t)\theta\bar{\theta}, \tag{33}$$

where $D_\theta = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}$ and $D_{\bar{\theta}} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}$ are the supercovariant derivatives of the $n = 2$ supersymmetry, $a(t)$ is a local time reparametrization parameter, $\beta'(t) = N^{-1/2}\beta$ is the Grassmann complex parameter of the local conformal $n = 2$ supersymmetry transformations and $b(t)$ is the parameter of the local $U(1)$ rotations on the Grassmann coordinates θ $(\bar{\theta} = \theta^\dagger)$. Then, the superfield generalization of the action (6), which is invariant under the $n = 2$ (LCTS) transformations (32) has the form [22, 23]

$$\begin{aligned}
S_{(n=2)} &= S_g + S_m = \int \left(-\frac{3}{\kappa^2} \mathbb{N}^{-1} \mathbb{R} D_{\bar{\theta}} \mathbb{R} D_{\theta} \mathbb{R} + \frac{3\sqrt{k}}{\kappa^2} \mathbb{R}^2 \right) d\theta d\bar{\theta} dt \\
&+ \int \left(\frac{1}{2} \mathbb{N}^{-1} \mathbb{R}^3 D_{\bar{\theta}} \mathbf{\Phi} D_{\theta} \mathbf{\Phi} - 2\mathbb{R}^3 g(\mathbf{\Phi}) \right) d\theta d\bar{\theta} dt,
\end{aligned} \tag{34}$$

where $g(\Phi)$ is the superpotential. The most general supersymmetric interaction for the set of complex homogeneous scalar fields with the scale factor was considered in [24, 25]. For the one-dimensional gravity superfield $\mathbb{N}(t,\theta,\bar{\theta})$ we have the following series expansion

$$\mathbb{N}(t,\theta,\bar{\theta}) = N(t) + i\theta\bar{\psi}'(t) + i\bar{\theta}\psi'(t) + V'(t)\theta\bar{\theta}, \tag{35}$$

where $N(t)$ is the lapse function, $\psi'(t) = N^{1/2}(t)\psi(t)$ and $V'(t) = N(t)V(t) + \bar{\psi}(t)\psi(t)$. The components $N, \psi, \bar{\psi}$ and V in (35) are gauge fields of the one-dimensional $n = 2$ supergravity. The superfield (35) transforms as the one-dimensional vector under the (LCTS) transformations (32),

$$\delta\mathbb{N} = (\mathbb{L}\mathbb{N})^{\cdot} + \frac{i}{2} D_{\bar{\theta}} \mathbb{L} D_{\theta} \mathbb{N} + \frac{i}{2} D_{\theta} \mathbb{L} D_{\bar{\theta}} \mathbb{N}. \tag{36}$$

The series expansion for the superfield $\mathbb{R}(t,\theta,\bar{\theta})$ has a similar form

$$\mathbb{R}(t,\theta,\bar{\theta}) = R(t) + i\theta\bar{\lambda}'(t) + i\bar{\theta}\lambda'(t) + B'(t)\theta\bar{\theta}, \tag{37}$$

where $R(t)$ is the scale factor of the FRW Universe, $\lambda' = \kappa N^{1/2}\lambda$ and $B'(t) = \kappa N(t)B(t) + \frac{\kappa}{2}(\bar\psi(t)\lambda(t) - \psi(t)\bar\lambda(t))$.

For the real scalar matter superfield $\mathbf{\Phi}(t,\theta,\bar\theta)$ we have

$$\mathbf{\Phi}(t,\theta,\bar\theta) = \phi(t) + i\theta\bar\chi'(t) + i\bar\theta\chi'(t) + F'(t)\theta\bar\theta, \tag{38}$$

where $\chi'(t) = N^{1/2}(t)\chi(t)$ and $F'(t) = N(t)F(t) + \frac{1}{2}(\bar\psi(t)\bar\chi(t) - \psi(t)\chi(t))$. The components $B(t)$ and $F(t)$ in the superfields $I\!R$ and $\mathbf{\Phi}$ are auxiliary fields. The superfields (37) and (38) transform as scalars under the (LCTS) transformations (32).

Performing the integration over $\theta,\bar\theta$ in (34) and eliminating the auxiliary fields B and F by means of their equations of motion, the action (34) takes its component form. The first-class constraints may be obtained from the component form of the action (34) varying it with respect to $N(t), \psi(t), \bar\psi(t)$ and $V(t)$, respectively. Then, we obtain the following first-class constraints $H_0 = 0$, $S = 0$, $\bar S = 0$ and $F = 0$, where

$$\begin{aligned} H_0 &= -\frac{\kappa^2}{12}\frac{\pi_R^2}{R} - \frac{3kR}{\kappa^2} - \frac{\sqrt{k}}{3R}\bar\lambda\lambda + \frac{\pi_\phi^2}{2R^3} - \frac{i\kappa}{2R^3}\pi_\phi(\bar\lambda\chi + \lambda\bar\chi) - \frac{\kappa^2}{4R^3}\bar\lambda\lambda\bar\chi\chi \\ &+ \frac{3\sqrt{k}}{2R}\bar\chi\chi + \kappa^2 g(\phi)\bar\lambda\lambda + 6\sqrt{k}g(\phi)R^2 + 2\left(\frac{\partial g}{\partial\phi}\right)^2 R^3 - 3\kappa^2 g^2(\phi)R^3 \\ &+ \frac{3}{2}\kappa^2 g(\phi)\bar\chi\chi + 2\frac{\partial^2 g}{\partial\phi^2}\bar\chi\chi + \kappa\frac{\partial g}{\partial\phi}(\bar\lambda\chi - \lambda\bar\chi), \end{aligned} \tag{39}$$

$$\begin{aligned} S &= \left(\frac{i\kappa}{3}R^{-1/2}\pi_R - \frac{2\sqrt{k}}{\kappa}R^{1/2} + 2\kappa g(\phi)R^{3/2} + \frac{\kappa}{4}R^{-3/2}\bar\chi\chi\right)\lambda \\ &+ \left(iR^{-3/2}\pi_\phi + 2R^{3/2}\frac{\partial g}{\partial\phi}\right)\chi, \\ \bar S &= S^\dagger, \end{aligned} \tag{40}$$

and

$$F = -\frac{2}{3}\bar\lambda\lambda + \bar\chi\chi, \tag{41}$$

The canonical Hamiltonian is the sum of all the constraints

$$H_{c(n=2)} = NH_0 + \frac{1}{2}\bar\psi S - \frac{1}{2}\psi\bar S + \frac{1}{2}VF. \tag{42}$$

In terms of Dirac's brackets for the canonical variables $R, \pi_R, \phi, \pi_\phi, \lambda, \bar\lambda, \chi$ and $\bar\chi$ the quantities $H_0, S, \bar S$ and F form the closed super-algebra of conserving charges

$$\begin{aligned} \{S,\bar S\}^* &= -2iH_0, \qquad \{H_0,S\}^* = \{H_0,\bar S\}^* = 0 \\ \{F,S\}^* &= iS, \qquad \{F,\bar S\}^* = -i\bar S. \end{aligned} \tag{43}$$

So, any physically allowed states must obey the following quantum constraints

$$H_0\psi = 0, \qquad S\psi = 0, \qquad \bar{S}\psi = 0, \qquad F\psi = 0, \tag{44}$$

when we change the classical variables by their corresponding operators. The first equation in (44) is the Wheeler-DeWitt equation for the minisuperspace model. Therefore, we have the *time-independent* Schrödinger equation, this fact is due to the invariance of the action (34) under reparametrization symmetry, this problem is well-known as the "problem of time" [1] in the minisuperspace models and general relativity theory. Due to the super-algebra (43) the second and the thirth equations in (44) reflect the fact, that there is a "square root" of the Hamiltonian H_0 with zero energy states. The constraints Hamiltonian H_0, supercharges $S, \bar{S}$ and the fermion number operator F follow from the invariance of the action (34) under the $n = 2$ (LCTS) transformations (32).

In order to have a time-dependent Schrödinger equation for the supersymmetric minisuperspace models with the action (34) we consider a generalization of the reparametrization invariant action S_r (14). In the case of $n = 2$ (LCTS) it has the superfield form

$$S_{r(n=2)} = \int \left[I\!\!P - \frac{i}{2} I\!\!N^{-1} \left(D_{\bar{\theta}} \mathbf{T} D_\theta I\!\!P - D_{\bar{\theta}} I\!\!P D_\theta \mathbf{T} \right) \right] d\theta d\bar{\theta} dt. \tag{45}$$

Note, that the $Ber E_B^A$, as well as the superjacobian of $n = 2$ (LCTS) transformations, is equal to one and is omitted in the actions (34,45). The action (45) is determined in terms of the new superfields $\mathbf{T}$ and $I\!\!P$. The series expansion for $\mathbf{T}$ has the form

$$\mathbf{T}(t, \theta, \bar{\theta}) = T(t) + \theta \eta'(t) - \bar{\theta} \bar{\eta}'(t) + m'(t) \theta \bar{\theta}, \tag{46}$$

where $\eta'(t) = N^{1/2}(t)\eta(t)$ and $m'(t) = N(t)m(t) + \frac{i}{2}(\bar{\psi}(t)\bar{\eta}(t) + \psi(t)\eta(t))$. The superfield $\mathbf{T}$ is determined by the odd complex time variables $\eta(t)$ and $\bar{\eta}(t)$, which are the superpartners of the time $T(t)$ and one auxiliary parameter $m(t)$. The transformation rule for the superfield $\mathbf{T}(t, \theta, \bar{\theta})$ under the $n = 2$ (LCTS) transformations (32) is

$$\delta \mathbf{T} = I\!\!L \dot{\mathbf{T}} + \frac{i}{2} D_{\bar{\theta}} I\!\!L D_\theta \mathbf{T} + \frac{i}{2} D_\theta I\!\!L D_{\bar{\theta}} \mathbf{T}. \tag{47}$$

The superfield $I\!\!P(t, \theta, \bar{\theta})$ has the form

$$I\!\!P(t, \theta, \bar{\theta}) = \rho(t) + i\theta P'_{\bar{\eta}}(t) + i\bar{\theta} P'_\eta(t) + P'_T(t) \theta \bar{\theta}, \tag{48}$$

where $P'_\eta(t) = N^{1/2} P_\eta$ and $P'_T(t) = N P_T + \frac{1}{2}(\bar{\psi} P_\eta - \psi P_{\bar{\eta}})$, P_η and $P_{\bar{\eta}}$ are the odd complex momenta, *i.e.* the superpartners of the momentum P_T.

The superfield $I\!\!P(t, \theta, \bar{\theta})$ transforms as

$$\delta I\!\!P(t, \theta, \bar{\theta}) = I\!\!L \dot{I\!\!P} + \frac{i}{2} D_{\bar{\theta}} I\!\!L D_\theta I\!\!P + \frac{i}{2} D_\theta I\!\!L D_{\bar{\theta}} I\!\!P. \tag{49}$$

The action (45) is invariant under the $n = 2$ (LCTS) transformations (32). Performing the integration over θ and $\bar{\theta}$ in (45) and making the redefinitions $P_T \to \frac{R^3}{\kappa^3} P_T$, $P_\eta \to \frac{R^3}{\kappa^3} P_\eta$ and $P_{\bar{\eta}} \to \frac{R^3}{\kappa^3} P_{\bar{\eta}}$ we obtain its component form

$$\begin{aligned} S_{r(n=2)} &= -\int \Bigg\{ \frac{R^3}{\kappa^3} \Bigg(P_T(N - \dot{T}) + i\dot{\eta}P_\eta + i\dot{\bar{\eta}}P_{\bar{\eta}} + \frac{\bar{\psi}}{2}(P_\eta - \bar{\eta}P_T) \\ &- \frac{\psi}{2}(P_{\bar{\eta}} - \eta P_T) + \frac{V}{2}(\eta P_\eta - \bar{\eta}P_{\bar{\eta}}) \Bigg) + m\dot{\rho} - \frac{iR^3}{2\kappa^3} m\psi P_{\bar{\eta}} - \frac{iR^3}{2\kappa^3} m\bar{\psi} P_\eta \Bigg\} dt. \end{aligned} \tag{50}$$

We can see from (50) that the momenta P_η, $P_{\bar{\eta}}$ and P_T in the superfield (48) are related with the components of the superfield (35), which enter in the action (34), unlike those momenta, the component ρ of the superfield (48) is not related with any components in (35). Therefore, the variables ρ and m can be eliminated from the action (50) by means of their equations of motion, then the component action has the final form

$$\begin{aligned} S_{r(n=2)} &= -\int \frac{R^3}{\kappa^3} \Bigg\{ P_T(N - \dot{T}) + i\dot{\eta}P_\eta + i\dot{\bar{\eta}}P_{\bar{\eta}} + \frac{\bar{\psi}}{2}(P_\eta - \bar{\eta}P_T) \\ &- \frac{\psi}{2}(P_{\bar{\eta}} - \eta P_T) + \frac{V}{2}(\eta P_\eta - \bar{\eta}P_{\bar{\eta}}) \Bigg\} dt. \end{aligned} \tag{51}$$

In addition to the canonical momenta π_T and π_{P_T} for the two even constraints (17), the action (51) has the additional momenta $\mathcal{P}_\eta$ and $\mathcal{P}_{P_\eta}$ conjugate to η and P_η, respectively,

$$\mathcal{P}_\eta = \frac{\partial L_{r(n=2)}}{\partial \dot{\eta}} = -i\frac{R^3}{\kappa^3} P_\eta, \qquad \mathcal{P}_{P_\eta} = \frac{\partial L_{r(n=2)}}{\partial \dot{P}_\eta} = 0. \tag{52}$$

With respect to the canonical odd Poisson brackets we have

$$\{\eta, \mathcal{P}_\eta\} = 1, \qquad \{P_\eta, \mathcal{P}_{P_\eta}\} = 1. \tag{53}$$

They form two primary constraints of second-class

$$\Pi_3(\eta) \equiv \mathcal{P}_\eta + i\frac{R^3}{\kappa^3} P_\eta = 0, \qquad \Pi_4(P_\eta) \equiv \mathcal{P}_{P_\eta} = 0. \tag{54}$$

The only non-vanishing Poisson bracket between these constraints is

$$\{\Pi_3, \Pi_4\} = i\frac{R^3}{\kappa^3}. \tag{55}$$

The momenta $\mathcal{P}_{\bar{\eta}}$ and $\mathcal{P}_{P_{\bar{\eta}}}$ conjugate to $\bar{\eta}$ and $P_{\bar{\eta}}$ respectively, also give two primary constraints of second-class

$$\Pi_5(\bar{\eta}) \equiv \mathcal{P}_{\bar{\eta}} + i\frac{R^3}{\kappa^3} P_{\bar{\eta}} = 0, \qquad \Pi_6(P_{\bar{\eta}}) = \mathcal{P}_{P_{\bar{\eta}}} = 0, \tag{56}$$

with non-vanishing Poisson bracket

$$\{\Pi_5, \Pi_6\} = i\frac{R^3}{\kappa^3}. \tag{57}$$

The constraints (54) and (56) for the Grassmann dynamical variables can be eliminated by Dirac's procedure. Defining the matrix constraint $C_{ik}(i,k=\eta, P_\eta, \bar\eta, P_{\bar\eta})$ as the odd Poisson bracket we have the following non-zero matrix elements

$$\begin{aligned} C_{\eta P_\eta} &= C_{P_\eta \eta} = \{\Pi_3, \Pi_4\} = i\frac{R^3}{\kappa^3}, \\ C_{\bar\eta P_{\bar\eta}} &= C_{P_{\bar\eta}\bar\eta} = \{\Pi_5, \Pi_6\} = i\frac{R^3}{\kappa^3}, \end{aligned} \tag{58}$$

with their inverse matrices $(C^{-1})^{\eta P_\eta} = -i\frac{\kappa^3}{R^3}$ and $(C^{-1})^{\bar\eta P_\eta} = -i\frac{\kappa^3}{R^3}$. The result of this procedure is the elimination of the momenta conjugate to the Grassmann variables, leaving us with the following non-zero Dirac's bracket relations

$$\{\eta, P_\eta\}^* = i\frac{\kappa^3}{R^3}, \qquad \{\bar\eta, P_{\bar\eta}\}^* = i\frac{\kappa^3}{R^3}. \tag{59}$$

So, if we take the additional term (45), then the full action is

$$\tilde S_{(n=2)} = S_{(n=2)} + S_{r(n=2)}. \tag{60}$$

The canonical Hamiltonian for the action (60) will have the following form

$$\begin{aligned} \tilde H_{c(n=2)} &= N\left(\frac{R^3}{\kappa^3}P_T + H_0\right) + \frac{\bar\psi}{2}\left(\frac{R^3}{\kappa^3}S_\eta + S\right) \\ &- \frac{\psi}{2}\left(-\frac{R^3}{\kappa^3}S_{\bar\eta} + \bar S\right) + \frac{V}{2}\left(\frac{R^3}{\kappa^3}F_\eta + F\right), \end{aligned} \tag{61}$$

where $S_\eta = (P_\eta - \bar\eta P_T)$, $S_{\bar\eta} = (-P_{\bar\eta} + \eta P_T)$, $F_\eta = (\eta P_\eta - \bar\eta P_{\bar\eta})$, and $H_0, S, \bar S$ and F are defined in (39,40,41). In the component form of the action (60) there are no kinetic terms for $N, \psi, \bar\psi$ and V. This fact is reflected in the primary constraints $P_N = 0$, $P_\psi = 0$, $P_{\bar\psi} = 0$ and $P_V = 0$, where $P_N, P_\psi, P_{\bar\psi}$ and P_V are the canonical momenta conjugate to $N, \psi, \bar\psi$ and V, respectively. Then, the total Hamiltonian may be written as

$$\tilde H = \tilde H_{c(n=2)} + u_N P_N + u_\psi P_\psi + u_{\bar\psi} P_{\bar\psi} + u_V P_V. \tag{62}$$

Due to the conditions $\dot P_N = \dot P_\psi = \dot P_{\bar\psi} = \dot P_V = 0$ we now have the first-class constraints

$$\begin{aligned} \tilde H &= \frac{R^3}{\kappa^3}P_T + H_0 = 0, \qquad \mathcal{F} = \frac{R^3}{\kappa^3}F_\eta + F = 0, \\ Q_\eta &= \frac{R^3}{\kappa^3}S_\eta + S = 0, \qquad Q_{\bar\eta} = -\frac{R^3}{\kappa^3}S_{\bar\eta} + \bar S = 0. \end{aligned} \tag{63}$$

They form a closed super-algebra with respect to the Dirac's brackets

$$\begin{aligned}\{Q_\eta, Q_{\bar\eta}\}^* &= -2i\tilde{H}, \qquad \{\tilde{H}, Q_\eta\}^* = \{\tilde{H}, Q_{\bar\eta}\}^* = 0 \\ \{\mathcal{F}, Q_\eta\}^* &= iQ_\eta, \qquad \{\mathcal{F}, Q_{\bar\eta}\}^* = -iQ_{\bar\eta}.\end{aligned} \tag{64}$$

After quantization Dirac's brackets must be replaced by anticommutators

$$\{\eta, P_\eta\} = i\{\eta, P_\eta\}^* = -\frac{\kappa^3}{R^3}, \qquad \{\bar\eta, P_{\bar\eta}\} = i\{\bar\eta, P_{\bar\eta}\}^* = -\frac{\kappa^3}{R^3}, \tag{65}$$

with the operator representation

$$P_\eta = -\frac{\kappa^3}{R^3}\frac{\partial}{\partial\eta}, \qquad P_{\bar\eta} = -\frac{\kappa^3}{R^3}\frac{\partial}{\partial\bar\eta}. \tag{66}$$

To obtain the quantum expression for $H_0, S, \bar{S}, F$ we must solve the operator ordering ambiguity. Such ambiguities always take place when the operator expression contains the product of non-commuting operators λ and $\bar\lambda$, χ and $\bar\chi$, R and $\pi_R = -i\frac{\partial}{\partial R}$, ϕ and $\pi_\phi = -i\frac{\partial}{\partial\phi}$. Such procedure leads in our case to the following expressions for the generators on the quantum level

$$\begin{aligned}\tilde{H} &= -i\frac{\partial}{\partial T} + H_0(R, \pi_R, \phi, \pi_\phi, \lambda, \bar\lambda, \chi\bar\chi), \\ Q_\eta &= \Big(\frac{\partial}{\partial\eta} - i\bar\eta\frac{\partial}{\partial T}\Big) + S(R, \pi_R, \phi, \pi_\phi, \lambda, \chi), \\ Q_{\bar\eta} &= -\Big(-\frac{\partial}{\partial\bar\eta} + i\eta\frac{\partial}{\partial T}\Big) + \bar{S}(R, \pi_R, \phi, \pi_\phi, \bar\lambda, \bar\chi), \\ \mathcal{F} &= \Big(\eta\frac{\partial}{\partial\eta} - \bar\eta\frac{\partial}{\partial\bar\eta}\Big) + F(\lambda, \bar\lambda, \chi, \bar\chi),\end{aligned} \tag{67}$$

where $S_\eta = \frac{\partial}{\partial\eta} - i\bar\eta\frac{\partial}{\partial T}$ and $S_{\bar\eta} = -\frac{\partial}{\partial\bar\eta} + i\eta\frac{\partial}{\partial T}$ are the generators of the supertranslation, $P_T = -i\frac{\partial}{\partial T}$ is the ordinary time translation on the superspace with coordinates $(t, \eta, \bar\eta)$

$$\{S_\eta, S_{\bar\eta}\} = 2i\frac{\partial}{\partial T}, \tag{68}$$

and $F_\eta = \eta\frac{\partial}{\partial\eta} - \bar\eta\frac{\partial}{\partial\bar\eta}$ is the $U(1)$ generator of the rotation on the complex Grassmann coordinate $\eta(\bar\eta = \eta^\dagger)$. The algebra of the quantum generators of the conserving charges $H_0, S, \bar{S}, F$ is a closed super-algebra

$$\begin{aligned}\{S, \bar{S}\} &= 2H_0, \quad [S, H_0] = [\bar{S}, H_0] = [F, H_0] = 0, \\ S^2 = \bar{S}^2 &= 0, \qquad [F, S] = -S, \qquad [F, \bar{S}] = \bar{S}.\end{aligned} \tag{69}$$

We can see from Eqs. (64) and (67) that the operators $\tilde{H}, Q_\eta, Q_{\bar\eta}$ and $\mathcal{F}$ obey the same super-algebra (69)

$$\begin{aligned}\{Q_\eta, Q_{\bar\eta}\} &= 2\tilde{H}, \quad [Q_\eta, \tilde{H}] = [Q_{\bar\eta}, \tilde{H}] = [\mathcal{F}, \tilde{H}] = 0 \\ Q_\eta^2 &= Q_{\bar\eta}^2 = 0, \qquad [\mathcal{F}, Q_\eta] = -Q_\eta, \quad [\mathcal{F}, Q_{\bar\eta}] = Q_{\bar\eta}.\end{aligned} \tag{70}$$

In the quantum theory the first-class constraints (67) become conditions on the wave function Ψ, which has the superfield form

$$\begin{aligned}\Psi(T,\eta,\bar\eta,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi) &= \psi(T,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi)\\ &+ i\eta\xi(T,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi)+i\bar\eta\zeta(T,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi)\\ &+ \sigma(T,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi)\eta\bar\eta. \end{aligned} \tag{71}$$

So, we have the supersymmetric quantum constraints

$$\tilde H\Psi = 0, \qquad Q_\eta\Psi = 0, \qquad Q_{\bar\eta}\Psi = 0, \qquad \mathcal{F}\Psi = 0. \tag{72}$$

As a consequence of the algebra (70) the constraints

$$Q_\eta\Psi = 0, \qquad Q_{\bar\eta}\Psi = 0, \tag{73}$$

lead to the equation

$$\{Q_\eta, Q_{\bar\eta}\}\Psi = 2\tilde H\Psi = 0, \tag{74}$$

which is a time-dependent Schrödinger equation for the minisuperspace model.

The condition (74) leads to the following form for the wave function (71)

$$\psi_* = \psi - \eta(S\psi) - \bar\eta(\bar S\psi) + \frac{1}{2}(\bar S S - S\bar S)\psi\eta\bar\eta, \tag{75}$$

then $Q_\eta\psi_*$ has the following form

$$\begin{aligned} Q_\eta\psi_* &= \bar\eta\Big(-i\frac{d\psi}{dT} + \frac{1}{2}\{S,\bar S\}\psi\Big) + \\ &+ \eta\bar\eta S\Big(-i\frac{d\psi}{dT} + \frac{1}{2}\{S,\bar S\}\psi\Big) = 0, \end{aligned} \tag{76}$$

this is the standard Schrödinger equation and due to the relation $H_0 = \frac{1}{2}\{S,\bar S\}$ it may be written as

$$i\frac{\partial\psi}{\partial T} = H_0\psi, \tag{77}$$

where the wave function is $\psi(T,R,\phi,\lambda,\bar\lambda,\chi,\bar\chi)$. If we put in the Schrödinger equation (77) the condition of a stationary state given by $\frac{\partial\psi}{\partial T} = 0$, we will have that $H_0\psi = 0$ and due to the algebra (69) we obtain $S\psi = \bar S\psi = 0$ and the wave function ψ_* becomes ψ.

The next step is to consider the additional term (30) in the general relativity theory and its consequences in the canonical formalism.

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